Separation Transformation and New Exact Solutions for the (1+*N*)-Dimensional Triple Sine-Gordon Equation

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The separation transformation method is extended to the $(1+N)$-dimensional triple Sine-Gordon equation and a special type of implicitly exact solution for this equation is obtained. The exact solution contains an arbitrary function which may lead to abundant localized structures of the high-dimensional nonlinear wave equations. The separation transformation method in this paper can also be applied to other kinds of high-dimensional nonlinear wave equations.

Key words: Separation Transformation; Triple Sine-Gordon Equation; Implicitly Exact Solution.
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1. Introduction

It is well known that the Sine-Gordon-type equations [1 – 10], including one-dimensional single Sine-Gordon (SG) equation $u_{xx} - u_{tt} = a\sin(u)$, double Sine-Gordon equation

$$u_{xx} - u_{tt} = a\sin(u) + b\sin(2u), \tag{1}$$

and triple Sine-Gordon equation

$$u_{xx} - u_{tt} = a\sin(u) + b\sin(2u) + c\sin(3u) \tag{2}$$

are widely applied in physics and engineering. These equations are special cases of nonlinear wave equations. Especially, the single Sine-Gordon equation is Lax integrable and has nontrivial prolongation structures [1 – 3, 11].

The (1+1)-dimensional Sine-Gordon equation [1 – 3] was originally considered in the nineteenth century in the course of studying surfaces of constant negative curvature. This equation attracted a lot of attention in the 1970s due to the presence of soliton solutions [12 – 14]. The double SG equation arises in many physical systems like the spin dynamics in the B phase of superfluid ^{3}He [4], some features of the propagation of resonant ultrashort optical pulses through degenerate media [5], quasi-one-dimensional charge density wave condensate theories of the organic linear conductors like TTF-TCNQ [6], and nonlinear excitations in a compressible chain of XY dipoles under conditions of piezoelectric coupling macromolecules [7]. In the case of resonant fivefold degenerate media, propagation and creation of ultrashort optical pulses, the anisotropic magnetic liquids ^{3}HeA and ^{3}HeB at temperatures below the transition to the A phase at 2.6 mK, and the propagation of spin waves, one usually uses the single SG and the double SG model. However, in some other cases, one has to use the higher triple SG equation [8 – 10] which describes the propagation of strictly resonant sharp line optical pulses through unexcited absorbing media with Q(3) symmetry and the extrapolation to Q(J). In these cases the selection rules for the optical transitions are $\triangle J = 0$, $\triangle MJ = 0$, where J is an angular momentum quantum number. In the experiments carried out by Bullough and Caudrey [10] J is the hyperfine structure quantum number $F = 2$, and the transition is $2F + 1 = 5$-fold degenerate.

The $(1+N)$-dimensional Sine-Gordon-type equation [15, 16] appears in many physically relevant systems and its properties have been studied from both mathematical and physical viewpoints. Recently, Wang et al. [17] and Yan [18] introduced an effective separation transformation to study the various $(1+N)$-dimensional nonlinear systems and some wonderful results are obtained. In this paper, we will apply this separation transformation approach to the $(1+N)$-dimensional triple SG equation in the form

$$u_{tt}-\sum_{j=1}^{N}u_{x_jx_j}=\frac{1}{3}a\sin\left(\frac{u}{3}\right)+\frac{2}{3}b\sin\left(\frac{2u}{3}\right)+c\sin(u). \tag{3}$$

The rest of this paper is organized as follows. In Section 2, some special type of implicitly exact solutions for (3) are obtained by means of the separation transformation approach. Conclusions are presented in Section 3.

2. Separation Transformation and New Exact Solutions for Equation (3)

In this part, we extend the separation transformation method to the $(1+N)$-dimensional triple SG equation (3). The following proposition reduces (3) to a nonlinear ordinary differential equation (ODE) and a system of partial differential equations (PDEs) with a nonlinear PDE and a linear PDE.

Proposition. Function $u(x_1,x_2,\cdots,x_N;t)=U[w(x_1,x_2,\cdots,x_N;t)]$ is the solution of (3) if the function $U[w(x_1,x_2,\cdots,x_N;t)]$ solves the following system of nonlinear ODE:

$$\begin{aligned}kU''(w)&=k\frac{\partial^2U}{\partial w^2}\\&=\frac{1}{3}a\sin\left(\frac{U}{3}\right)+\frac{2}{3}b\sin\left(\frac{2U}{3}\right)+c\sin(U),\end{aligned} \tag{4}$$

and the function $w=w(x_1,x_2,\cdots,x_N;t)$ satisfies

$$\sum_{j=1}^{N}\left(\frac{\partial w}{\partial x_i}\right)^2-\left(\frac{\partial w}{\partial t}\right)^2=k,\quad \sum_{j=1}^{N}\frac{\partial^2 w}{\partial x_i^2}-\frac{\partial^2 w}{\partial t^2}=0, \tag{5}$$

where k is a nonzero constant.

The proof of this proposition is similar to that of References [17, 18].

From this proposition we find that (3) is now separated into two groups of differential equations, namely (4) and (5). Once we have the exact solutions of (4) and (5), the exact solutions of (3) are obtained immediately from the transformation $u(x_1,x_2,\cdots,x_N;t)=U[w(x_1,x_2,\cdots,x_N;t)]$.

We first solve the system of partial differential equations (5).

Case 1. When $N=1$, (5) becomes

$$\left(\frac{\partial w}{\partial x_1}\right)^2-\left(\frac{\partial w}{\partial t}\right)^2=k,\quad \frac{\partial^2 w}{\partial x_1^2}-\frac{\partial^2 w}{\partial t^2}=0$$

with

$$w(x_1,t)=c_1(x_1+t)+\frac{k}{4c_1}(x_1-t)+c_2, \tag{6}$$

where c_1 and c_2 are integral constants.

Case 2. When $N\geq 2$, (5) has the following solution

$$\begin{aligned}w(x_1,x_2,\cdots,x_N;t)&=h\left(\sum_{j=1}^{N}\lambda_jx_j+\varepsilon t+c_3\right)\\&+\frac{k}{\sqrt{\sum_{j=1}^{N}l_j^2-\delta^2}}\left(\sum_{j=1}^{N}l_jx_j+\delta t\right),\end{aligned} \tag{7}$$

$$\varepsilon\delta=\sum_{j=1}^{N}\lambda_jl_j,\quad \varepsilon^2=\sum_{j=1}^{N}\lambda_j^2, \tag{8}$$

where $h=h(x_1,\cdots,x_N;t)$ is an arbitrary function, λ_j, x_j, $(j=1,\cdots,N)$, ε, δ, c_3 are all constants.

In what follows, we solve the ODE (4) by using special transformations [19–25] and direct integral technique. Multiplying both sides of ODE (4) with U', we have

$$U'U''=\frac{U'}{k}\left[\frac{1}{3}a\sin\left(\frac{U}{3}\right)+\frac{2}{3}b\sin\left(\frac{2U}{3}\right)+c\sin(U)\right]. \tag{9}$$

Then integrating (9) and letting the integral constant be C_1 we arrive at

$$\begin{aligned}(U')^2=\frac{2}{k}\bigg[&-c\cos(U)-a\cos\left(\frac{U}{3}\right)\\&-b\cos\left(\frac{2U}{3}\right)\bigg]+C_1.\end{aligned} \tag{10}$$

In order to reduce (10) to a solvable equation, we introduce a basic transformation of field U as

$$U=3\arccos(v+A) \tag{11}$$

with v being a function of w and A being a constant. The substitution of (11) into (10) yields

$$\begin{aligned}(v')^2=&\frac{8}{9k}v^5+\left(\frac{40A}{9k}+\frac{4}{9k}\right)v^4\\&+\left(\frac{16A}{9k}-\frac{4}{3k}+\frac{80A^2}{9k}\right)v^3\end{aligned}$$

$$
+\left(-\frac{2}{3k}-\frac{4A}{k}+\frac{8A^2}{3k}+\frac{80A^3}{9k}-\frac{1}{9}C_1\right)v^2
+\left(\frac{4}{9k}-\frac{4A^2}{k}-\frac{4A}{3k}+\frac{16A^3}{9k}+\frac{40A^4}{9k}-\frac{2}{9}C_1A\right)v
+\frac{2}{9k}-\frac{4A^3}{3k}+\frac{4A}{9k}-\frac{2A^2}{3k}+\frac{8A^5}{9k}
+\frac{4A^4}{9K}-\frac{1}{9}C_1A^2,
$$

which can be further rewritten as

$$
(v')^2=\alpha_0+\alpha_1 v+\alpha_2 v^2+\alpha_3 v^3+\alpha_4 v^4+\alpha_5 v^5 \quad (12)
$$

with

$$
\begin{aligned}
\alpha_0&=\frac{2}{9k}-\frac{4A^3}{3k}+\frac{4A}{9k}-\frac{2A^2}{3k}+\frac{8A^5}{9k}+\frac{4A^4}{9K}-\frac{1}{9}C_1A^2,\\
\alpha_1&=\frac{4}{9k}-\frac{4A^2}{k}-\frac{4A}{3k}+\frac{16A^3}{9k}+\frac{40A^4}{9k}-\frac{2}{9}C_1A,\\
\alpha_2&=-\frac{2}{3k}-\frac{4A}{k}+\frac{8A^2}{3k}+\frac{80A^3}{9k}-\frac{1}{9}C_1,\\
\alpha_3&=\frac{16A}{9k}-\frac{4}{3k}+\frac{80A^2}{9k},\\
\alpha_4&=\frac{40A}{9k}+\frac{4}{9k},\quad \alpha_5=\frac{8}{9k}.
\end{aligned}
$$

According to the previous results on the ϕ^5 model [26] with some special parameters we can express the solutions of the five degree ODE (13) as

$$
v(w)=\beta_0+\beta_1 f(w)+\beta_2 f^2(w). \quad (13)
$$

Substituting (13) into (12), we want to transform (12) into

$$
\frac{\partial f}{\partial w}=d_0+d_1 f+d_2 f^2+d_3 f^3+d_4 f^4, \quad (14)
$$

so the parameters must be satisfied

$$
\begin{aligned}
\alpha_1&=\frac{4}{\beta_2^3}\left[5d_4^2\beta_0^4-8d_4\beta_0^3d_2\beta_2+6\beta_0d_0d_4\beta_2^2+d_0^2\beta_2^4-4\beta_0d_0d_2\beta_2^3\right],\\
\alpha_2&=\frac{-4}{\beta_2^3}\left[10d_4\beta_0^3-12d_4\beta_0^2d_2\beta_2+6\beta_0d_0d_4\beta_2^2+3\beta_0d_2^2\beta_2^2-2d_0d_2\beta_2^3\right],\\
\alpha_3&=\frac{4(10d_4^2\beta_0^2-8d_4\beta_0d_2\beta_2+2d_0d_4\beta_2^2+d_2^2\beta_2^2)}{\beta_2^2},\\
\alpha_4&=\frac{4d_4(-2d_2\beta_2+5d_4\beta_0)}{\beta_2^3},\\
a&=-\frac{9k(4d_2^2+4d_0d_2\beta_2-15d_0^2\beta_2^2)}{32\beta_2},\\
b&=\frac{9k(d_0\beta_2+2d_2)(2d_2+5d_0\beta_2)}{16\beta_2},\\
c&=\frac{9k(d_0\beta_2+2d_2)^2}{32\beta_2},\\
C_1&=\frac{9(4d_2^2+12d_0d_2\beta_2+13d_0^2\beta_2^2)}{8\beta_2},\\
\beta_1&=0,\quad \alpha_5=\frac{4d_4^2}{\beta_2^3},\quad d_1=d_3=0,\\
d_4&=-\frac{1}{4}\beta_2^2d_0-\frac{1}{2}d_2\beta_2,\quad A=-\beta_0-1,
\end{aligned}
$$

with β_0, β_2, d_0, d_2, k being arbitrary constants. From these relations, we find $f=f(w)$ satisfies

$$
\frac{\partial f}{\partial w}=d_0+d_2f^2+\left(-\frac{1}{4}{\beta_2}^2d_0-\frac{1}{2}d_2\beta_2\right)f^4, \quad (15)
$$

and $v=v(w)$ satisfies

$$
v=\beta_0+\beta_2 f^2. \quad (16)
$$

Solving (15) by direct integral we have its implicit exact solution as

$$
w-\frac{\sqrt{2}(d_0\beta_2+2d_2)\arctan\left(\frac{(d_0\beta_2+2d_2)f\sqrt{2}}{2\sqrt{d_0(d_0\beta_2+2d_2)}}\right)}{2(d_0\beta_2+d_2)\sqrt{d_0(d_0\beta_2+2d_2)}}
-\frac{\sqrt{\beta_2}\sqrt{d_2}\operatorname{arctanh}\left(\frac{\sqrt{2}}{2}f\sqrt{\beta_2}\right)}{2d_0\beta_2+2d_2}+C_1=0. \quad (17)
$$

Combining (16) with (11), the implicit exact solution for the $(1+N)$-dimensional triple SG equation (3) is obtained as

$$
u(x_1,x_2,\cdots,x_N;t)=3\arccos(\beta_2f^2-1), \quad (18)
$$

where $f=f(w)$ satisfies (17) and $w=w(x_1,x_2,\cdots,x_N;t)$ are determined by (6)–(8).

In what follows, we choose special parameters d_0, d_2, β_2 to analyze the physical properties of our implicit exact solution.

(i) If the constants d_0, d_2, β_2 in (17) are taken as $d_0 = 1$, $d_2 = 2$, $\beta_2 = 4$, then the coefficients a, b, c of the $(1+N)$-dimensional triple SG equation (3) are

$$a = \frac{27}{2}k, \quad b = 27k, \quad c = \frac{9}{2}k, \tag{19}$$

and its implicit exact solution reads

$$u(x_1,x_2,\cdots,x_N;t) = 3\arccos(4f^2-1) \tag{20}$$

with $f = f(w)$ satisfying

$$w - \frac{1}{3}\arctan(2f) - \frac{\sqrt{2}}{6}\operatorname{arctanh}(\sqrt{2}f) + 90 = 0. \tag{21}$$

(ii) If the constant d_0, d_2, β_2 in (17) are taken as $d_0 = 1$, $d_2 = 7$, $\beta_2 = 4$, then the coefficients a, b, c of the $(1+N)$-dimensional triple SG equation (3) are

$$a = -\frac{153}{32}k, \quad b = \frac{1377}{16}k, \quad c = \frac{729}{32}k, \tag{22}$$

and its implicit exact solution is just (20) with $f = f(w)$ satisfying

$$w - \frac{3}{11}\arctan(3f) - \frac{\sqrt{2}}{11}\operatorname{arctanh}(\sqrt{2}f) + \frac{1665}{8} = 0. \tag{23}$$

(iii) If the constant d_0, d_2, β_2 in (17) are taken as $d_0 = -1$, $d_2 = 1$, $\beta_2 = 4$, then the coefficients a, b, c of the $(1+N)$-dimensional triple SG equation (3) are

$$a = \frac{567}{32}k, \quad b = \frac{81}{16}k, \quad c = \frac{9}{32}k, \tag{24}$$

and its implicit exact solution is just (20) with $f = f(w)$ satisfying

$$w + \frac{1}{3}\arctan(f) + \frac{\sqrt{2}}{3}\operatorname{arctanh}(\sqrt{2}f) + \frac{369}{8} = 0. \tag{25}$$

(iv) If the constant d_0, d_2, β_2 in (17) are taken as $d_0 = -1$, $d_2 = -7$, $\beta_2 = 4$, then the coefficients a, b, c of the $(1+N)$-dimensional triple SG equation (3) are

$$a = -\frac{153}{32}k, \quad b = \frac{1377}{16}k, \quad c = \frac{729}{32}k, \tag{26}$$

and its implicit exact solution is just (20) with $f = f(w)$ satisfying

$$w + \frac{3}{11}\arctan(3f) + \frac{\sqrt{2}}{11}\operatorname{arctanh}(\sqrt{2}f) + \frac{1665}{8} = 0. \tag{27}$$

When $N = 1$, (3) becomes the (1+1)-dimensional triple SG equation

$$u_{tt} - u_{xx} = \frac{1}{3}a\sin\left(\frac{u}{3}\right) + \frac{2}{3}b\sin\left(\frac{2u}{3}\right) + c\sin(u) \tag{28}$$

by replacing x_1 with x. In this case, we have $w = c_1(x+t) + \frac{k}{4c_1}(x-t) + c_2$. Therefore, for the coefficients a, b, c in (19), (22), (24), (26) the exact solutions of the (1+1)-dimensional triple SG equation (28) is (20) with f described by (21), (23), (25), (27), respectively.

When $N = 2$, we arrive at the (1+2)-dimensional triple SG equation

$$u_{tt} - u_{xx} - u_{yy} = \frac{1}{3}a\sin\left(\frac{u}{3}\right) + \frac{2}{3}b\sin\left(\frac{2u}{3}\right) + c\sin(u) \tag{29}$$

by replacing x_1, x_2 with x, y, respectively. In this case, we have

$$\begin{aligned} w &= h(\lambda_1 x + \lambda_2 y + \varepsilon t + c_3) \\ &\quad + \frac{k}{\sqrt{l_1^2 + l_2^2 - \delta^2}}(l_1 x + l_2 y + \delta t), \end{aligned} \tag{30}$$

where $h(\cdot)$ is an arbitrary function, and $\varepsilon\delta = \lambda_1 l_1 + \lambda_2 l_2$, $\varepsilon^2 = \lambda_1^2 + \lambda_2^2$. Therefore, for the coefficients a, b, c in (19), (22), (24), (26) the exact solutions of the (1+2)-dimensional triple SG equation (29) is (20) with w described by (30) and f described by (21), (23), (25), (27), respectively.

When $N = 3$, we arrive at the (1+3)-dimensional triple SG equation

$$\begin{aligned} &u_{tt} - u_{xx} - u_{yy} - u_{zz} = \\ &\quad \frac{1}{3}a\sin\left(\frac{u}{3}\right) + \frac{2}{3}b\sin\left(\frac{2u}{3}\right) + c\sin(u) \end{aligned} \tag{31}$$

by replacing x_1, x_2, x_3 with x, y, z, respectively. In this case, we have

$$\begin{aligned} w &= h(\lambda_1 x + \lambda_2 y + \lambda_3 z + \varepsilon t + c_3) \\ &\quad + \frac{k}{\sqrt{l_1^2 + l_2^2 + l_3^2 - \delta^2}}(l_1 x + l_2 y + l_3 z + \delta t), \end{aligned} \tag{32}$$

where $h(\cdot)$ is an arbitrary function, and $\varepsilon\delta = \lambda_1 l_1 + \lambda_2 l_2 + \lambda_3 l_3$, $\varepsilon^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$. Therefore, for the coefficients a, b, c in (20), (23), (25), (27) the exact solutions of the (1+3)-dimensional triple SG equation (32) is (21) with w described by (33) and f described by (22), (24), (26), (28), respectively.

Remark. We have extended the separation transformation approach to the $(1+N)$-dimensional triple Sine-Gordon equation and derived the implicitly exact solutions of this equation. It has been shown that when $N > 1$, there is an arbitrary function $h = h\left(\sum_{j=1}^{N} \lambda_j x_j + \varepsilon t + c_3\right)$ in the implicitly exact solution of the $(1+N)$-dimensional triple Sine-Gordon equation, which maybe lead to abundant localized structures [27, 28]. We conclude that although most high-dimensional nonlinear equations are non-integrable, they may have rich localized nonlinear structures.

3. Conclusion

In conclusion, a special type of implicitly exact solution with an arbitrary function for the $(1+N)$-dimensional triple Sine-Gordon equation is proposed by extending the separation transformation method. The special exact solution may be useful to explain certain localized nonlinear phenomena in some scientific fields. Especially, the result in this paper can be applied to describe the propagation of strictly resonant sharp line optical pulses in the near future.

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